JOURNAL OF ALGEBRA 113, 339–345 (1988)

Complexes and Coxeter Groups—Operations and Outer Automorphisms

LYNNE D. JAMES*

School of Mathematics, The University, Newcastle upon Tyne NE1 7RU, England

Communicated by Jacques Tits

Received February 19, 1986

1. INTRODUCTION

There is a well-known duality for maps on surfaces that interchanges vertices and faces while retaining certain important features such as the automorphism group. Wilson [12] and Lins [7] gave topological descriptions for other similar invertible operations on surface maps which, together with the above duality, generate a group isomorphic to S_3. In [6] Jones and Thornton showed how these operations arise naturally in algebraic map theory, being induced by the outer automorphisms of a certain Coxeter group. This paper presents this result in general dimension.

2. COMPLEXES AND COXETER GROUPS

In [9] Ronan shows how any chamber system, as defined by Tits [10], can be regarded as a certain type of cell complex. Special cases of the connection between thin chamber systems and cell complexes arise as (essentially) algebraic approaches to cell decompositions of manifolds and their natural generalisations (see [2–4, 11], for example). We briefly outline the connection between certain Coxeter groups and ordered complexes.

Suppose that we have a cell decomposition of a connected n-manifold. We take a barycentric subdivision and label the vertices of this subdivision with the dimension of the represented cell. We now form a dual graph whose edges are labelled by elements of $\{0, 1, ..., n\}$ such that no two incident edges have the same label. This is done by regarding the n-simplices of the barycentric subdivision as vertices which share an i-labelled edge whenever there is a common $(n-1)$-face of the barycentric subdivision that does not contain a vertex labelled i.

* Current address: Faculty of Mathematical Studies, The University, Southampton SO9 5NH, England.

0021-8693/88 $3.00

For each integer i: $0 \leqslant i \leqslant n$ we define ρ_i to be the permutation of the vertices of this graph that transposes the vertices of each edge labelled i. It can be shown that the relations $\rho_i^2 = (\rho_j \rho_k)^2 = 1$ hold for $|j-k| > 1$, and so, by the connectivity of the manifold, we have a transitive permutation representation of the group

$$\Gamma_n = \langle r_0, r_1, ..., r_n \mid r_i^2 = (r_j r_k)^2 = 1, 0 \leqslant i, j, k \leqslant n, |j-k| > 1 \rangle.$$

Conversely, given such a transitive permutation representation we can reconstruct the manifold and its cell decomposition. Roughly speaking we take a set of n-simplices, one for each element of the permuted set, and arbitrarily label each vertex of each n-simplex with a distinct element of $\{0, 1, ..., n\}$. We then identify two $(n-1)$-faces, that do not contain a vertex labelled i, whenever the corresponding elements are transposed by r_i.

If we take an arbitrary transitive permutation representation of Γ_n then the underlying space of the associated complex may not be a manifold. However, there is a partial order on a set of subcomplexes of the associated complex which, when the associated complex is a barycentric subdivision of a cell decomposition of a connected manifold, is the natural partial order (by inclusion) on the cells. Transitive permutation representations of Γ_n characterise such ordered complexes (see [11]). Details on the representation of cell decompositions of manifolds by means of Γ_n—sets may be found in [3].

3. Operations and Outer Automorphisms

Suppose that $\mathscr{M}$ is a cell decomposition of a connected n-manifold. Let $\mathscr{M}^*$ denote the dual decomposition. Cells of dimension i in $\mathscr{M}$ are transformed into cells of dimension $n-i$ in $\mathscr{M}^*$. Let $\Delta\mathscr{M}$ denote a barycentric decomposition of $\mathscr{M}$. If we consider the relationship between paths in $\mathscr{M}$ along the graph dual to $\Delta\mathscr{M}$ and paths in $\mathscr{M}^*$ along the graph dual to $\Delta\mathscr{M}^*$ then we see that the duality of cell decompositions of manifolds corresponds to a duality of sequences in $\{r_0, ..., r_n\}$ that acts by interchanging the symbols r_i and r_{n-i}. This duality preserves the set of sequences $\{r_i^2, (r_j r_k)^2$: $0 \leqslant i, j, k \leqslant n, |j-k| > 1\}$. It also preserves the juxtaposition of sequences. Thus, we have a duality of elements in Γ_n that preserves group multiplication. In other words, we have a group automorphism of Γ_n. This group automorphism induces the duality of cell decompositions of manifolds by its action (pre-composition) on the associated transitive permutation representations of Γ_n. We define an *operation* on the topological realisations of transitive permutation representations of Γ_n to be any transformation that is induced by a group automorphism of Γ_n. This definition was first made by Jones and Thornton [6] for maps on surfaces.

The inner automorphisms of Γ_n act trivially on the topological realisations of transitive permutation representations of Γ_n, so we have an induced action of the group of outer automorphisms of Γ_n:

$$\mathrm{Out}(\Gamma_n) \cong \mathrm{Aut}(\Gamma_n)/\mathrm{Inn}(\Gamma_n).$$

We now come to the main theorem.

THEOREM. *If H_n is the subgroup of automorphisms of Γ_n that is generated by θ_n: $r_i \mapsto r_{n-i}$, $0 \leqslant i \leqslant n$, and ϕ_n: $r_2 \mapsto r_0 r_2$; $r_i \mapsto r_i$, $0 \leqslant i \leqslant n$, $i \neq 2$, then* $\mathrm{Aut}(\Gamma_n)$ *is a split extension of* $\mathrm{Inn}(\Gamma_n)$ *by H_n.*

The theorem was proved in the case $n = 2$ in [6]. It is easily verified that H_2 is isomorphic to the symmetric group S_3. For $n > 2$, H_n is dihedral of order 8.

4. CENTRALISERS OF INVOLUTIONS

Our aim is to prove the theorem stated in Section 3.

First a word about Coxeter groups. Let W be any group generated by a set of involutions S. For s, s' in S let $m(s, s')$ be the order of ss' and let I be the set of couples (s, s') such that $m(s, s')$ is finite. The couple (W, S) is called a *Coxeter system* if the generating set S and the relations $(ss')^{m(s,s')} = 1$ for (s, s') in I forms a presentation of the group W.

By considering surjections from Γ_n onto dihedral groups we see that if $|j-k| > 1$ then $r_j r_k$ has order 2 and if $|j-k| = 1$ then $r_j r_k$ is of infinite order. Thus if R_n denotes the subset $\{r_0, r_1, ..., r_n\}$ then the couple (Γ_n, R_n) is a Coxeter system.

THEOREM 4.1 [1, Theorem IV.1.8]. *If (W, S) is a Coxeter system then*

(i) *For all subsets X of S, the couple $(\langle X \rangle, X)$ is a Coxeter system*;

(ii) *If $(X_\alpha)_{\alpha \in A}$ is a family of subsets of S then* $\langle \bigcap_{\alpha \in A} X_\alpha \rangle = \bigcap_{\alpha \in A} \langle X_\alpha \rangle$.

Any Coxeter system (W, S) may be represented by an edge-labelled graph. The vertices are taken to be the elements of S, and two vertices s, s' are joined by an edge labelled $m(s, s')$ whenever $m(s, s') > 2$.

Thus (Γ_n, R_n) is represented by the edge-labelled graph shown in Fig. 1.

The graph representing (Γ_n, R_n) is obtained from the graph representing (Γ_{n-1}, R_{n-1}) by the addition of a single vertex, representing r_n, and an edge labelled ∞. This suggests that we might use induction on n to prove

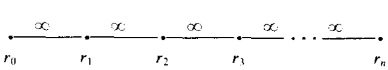

FIG. 1. The graph representing (Γ_n, R_n).

481/113/2-6

the theorem stated in Section 3 if we first determine the behaviour of r_n under the automorphism group of Γ_n. Our immediate aim is thus to study some simple properties of r_n that remain invariant under the automorphism group of Γ_n. We consider the order of r_n and the isomorphism class of its centraliser.

For each $i \in \{0, 1, ..., n\}$ let $G_1(n, r_i)$ denote the subgroup of Γ_n generated by $R_n \backslash \{r_i\}$, let $G_2(n, r_i)$ denote the subgroup of Γ_n generated by $R_n \backslash \{r_{i \pm 1}\}$, and let $A(n, r_i)$ denote the subgroup of Γ_n generated by $R_n \backslash \{r_i, r_{i \pm 1}\}$. Then Γ_n is the amalgamated product $P(n, r_i)$ of the subgroups $G_1(n, r_i)$ and $G_2(n, r_i)$ by the subgroup $A(n, r_i)$.

PROPOSITION 4.2. *There is a bijection between the sets of pairwise commuting elements of R_n and the conjugacy classes of elements of finite order in Γ_n.*

Proof. We use induction on n. Let T be the set of those $n \in \mathbb{N}$ for which any element of finite order in Γ_n is conjugate to a product of pairwise commuting elements of R_n. Clearly $0, 1 \in T$. Suppose that $k \geqslant 2$ and that $n \in T$ for all $n < k$. By the torsion theorem for amalgamated products [8, Theorem IV.2.7], any element of finite order in Γ_k is conjugate to an element of either $G_1(k, r_k)$ or $G_2(k, r_k)$. But $G_1(k, r_k) = \Gamma_{k-1}$ and $G_2(k, r_k) = \Gamma_{k-2} \times \langle r_k \rangle$, and so, by the inductive hypothesis, $k \in T$. Thus $T = \mathbb{N}$.

By abelianisation we obtain a surjection from Γ_n onto the direct product of $n+1$ copies of a cyclic group of order 2. This shows that if the product of one set of pairwise commuting elements of R_n is conjugate to the product of another then those two sets must be equal. ∎

PROPOSITION 4.3. *The centraliser in Γ_n of the product of a set of pairwise commuting elements of R_n is the subgroup associated with the edge-labelled graph representing (Γ_n, R_n) less those vertices adjacent to some element of that set.*

Proof. Let $\{s_1, ..., s_m\}$ be a set of pairwise commuting elements of R_n and let $s = s_1 \cdots s_m$. Let $i \in \{1, ..., m\}$. Then $s \in G_2(n, s_i) \backslash A(n, s_i)$. Suppose that $t \in C_{\Gamma_n}(s)$. Let $t = t_1 \cdots t_k$ be a reduced form for t in $P(n, s_i)$ and suppose that $k \geqslant 2$. If $t_1, t_k \in G_2(n, s_i)$ then $s^{t_i} \in G_2(n, s_i) \backslash A(n, s_i)$ and $t_k s \in A(n, s_i)$. Whence $t_{k-1} t_k s \in G_1(n, s_i) \backslash A(n, s_i)$ and so $t^{-1} s t s = t_k^{-1} \cdots t_2^{-1} \cdot s^{t_1} \cdot t_2 \cdots t_{k-2}(t_{k-1} t_k s) \neq 1$ by the normal form theorem for amalgamated products [8, Theorem IV.2.6]. If either t_1 or t_k is in $G_1(n, s_i)$ then we find a similar contradiction. Whence $k \leqslant 1$. If $t \in G_1(n, s_i) \backslash A(n, s_i)$ then again $t^{-1} s t s \neq 1$ by the normal form theorem for amalgamated products. Whence $t \in G_2(n, s_i)$. Therefore $C_{\Gamma_n}(s) \subseteq \bigcap_{i=1}^{m} G_2(n, s_i)$. Proposition 4.3 now follows as a corollary to Theorem 4.1. ∎

COROLLARY 4.4. *If $n>0$ then $Z(\Gamma_n)=1$.*

Proof. If $g \in Z(\Gamma_n)$ then

$$\begin{aligned} g &\in C_{\Gamma_n}(r_0 r_2 r_4 \cdots) \cap C_{\Gamma_n}(r_1 r_3 r_5 \cdots) \\ &= \langle r_0, r_2, r_4, \ldots \rangle \cap \langle r_1, r_3, r_5, \ldots \rangle \\ &= 1, \end{aligned}$$

by Theorem 4.1. ∎

Our aim now is to use Propositions 4.2 and 4.3 to determine the behaviour of r_n under the automorphism group of Γ_n in the sense that if $\alpha \in \mathrm{Aut}(\Gamma_n)$ then we will show that there exists $\alpha_1 \in \langle \Gamma_n, H_n \rangle$ such that $r_n^{\alpha\alpha_1} = r_n$. We make use of the following construction. For $g \in \Gamma_n$ consider those involutions $h \in \Gamma_n$ that commute with g and are such that $ZC_{\Gamma_n}(h) \cong C_2$. Let $\mu(g)$ be the number of conjugacy classes represented by such involutions. By Proposition 4.2, $\mu(g)$ is finite. Clearly, if $\alpha \in \mathrm{Aut}(\Gamma_n)$ then $\mu(g)=\mu(g^\alpha)$.

LEMMA 4.5. $\mu(r_i) = |\{r_j \in R_n : j \notin \{2, n-2, i \pm 1\}\}|$ *for all* $r_i \in R_n$.

Proof. By Propositions 4.2 and 4.3, $\{r_j \in R_n : j \notin \{2, n-2\}\}$ represents the conjugacy classes of those involutions $h \in \Gamma_n$ such that $ZC_{\Gamma_n}(h) \cong C_2$. If $|j-i|=1$ then $r_i \notin C_{\Gamma_n}(r_j)^{\Gamma_n}$, by abelianisation, and so the conjugacy class of r_j is not counted in $\mu(r_i)$. Conversely, if $|j-i| \neq 1$ and $j \notin \{2, n-2\}$ then the conjugacy class of r_j is counted in $\mu(r_i)$. ∎

COROLLARY 4.6. *If $\mu(r_i)=\mu(r_n)$ then $i \in \{0, 1, 3, n-3, n-1, n\}$.*

PROPOSITION 4.7. *Given $\alpha \in \mathrm{Aut}(\Gamma_n)$ there exists $\alpha_1 \in \langle \Gamma_n, H_n \rangle$ such that $r_n^{\alpha\alpha_1} = r_n$.*

Proof. If we consider the order of r_n and its centraliser then the cases $n \leqslant 2$ are trivial. For $n \geqslant 3$ we have $ZC_{\Gamma_n}(r_n) \cong C_2$ and so the conjugacy class of r_n^α is represented by some $r_i \in R_n$. Thus, by Corollary 4.6, there is some $g \in \Gamma_n$ and some $\delta \in \langle \theta_n \rangle$ such that $r_n^{\alpha g \delta} \in \{r_{n-3}, r_{n-1}, r_n\}$. If $r_n^{\alpha g \delta} = r_n$ then we are done. We deal separately with the remaining cases.

Let P_n denote the set of sets of pairwise commuting elements of R_n and, for $g \in \Gamma_n$, let $U(g) = C_{\Gamma_n}(g)/ZC_{\Gamma_n}(g)$. Suppose that $r_n^{\alpha g \delta} = r_{n-1}$. Then $U(r_n) \cong U(r_{n-1})$. If $n=3$, this implies the contradiction that $\Gamma_1 \cong 1$. If $n>3$, this implies that $\Gamma_{n-2} \cong \Gamma_{n-3}$, and thus (by Proposition 4.2) the contradiction that $|P_{n-2}| = |P_{n-3}|$.

On the other hand, suppose that $r_n^{\alpha g \delta} = r_{n-3}$. If $n=3$ then $r_n^{\alpha g \delta'} = r_n$, where $\{\delta, \delta'\} = \{1, \theta_n\}$, and we are done. If $n \in \{4, 5\}$ then $U(r_n) \cong U(r_{n-3})$

implies that $\Gamma_{n-2} \cong \Gamma_1$, and thus the contradiction that $|P_{n-2}| = |P_1|$. If $n>5$ then $\Gamma_{n-2} \cong \Gamma_{n-5} \times \Gamma_1$ and thus the contradiction that $|P_{n-2}| = 3\,|P_{n-5}|$, for if $S \in P_{n-5}$ then S, $S \cup \{r_{n-3}\}$, $S \cup \{r_{n-2}\}$, $\{r_{n-4}\} \in P_{n-2}$. ∎

We have determined the behaviour of r_n under the automorphism group of Γ_n. Our motivation for his was the hope of using induction on n to prove the theorem stated in Section 3. We now prove the theorem.

5. The Main Theorem

Our immediate aim is to develop the inductive step that will be used to prove the theorem stated in Section 3. In view of Proposition 4.7 we note that if $\rho \in \mathrm{Aut}(\Gamma_n)$ and $r_n^\rho = r_n$ then ρ restricts to an automorphism of $C_{\Gamma_n}(r_n)$, which, for $n \geqslant 2$, is $\Gamma_{n-2} \times \langle r_n \rangle$. The following simple proposition thus relates $\mathrm{Aut}(\Gamma_n)$ to $\mathrm{Aut}(\Gamma_{n-2})$, and so gives the necessary inductive step.

Proposition 5.1. *For any two groups G and H let π be the natural epimorphism $\pi\colon G \times H \to G$. If $\rho \in \mathrm{Aut}(G \times H)$ and $H^\rho = H$ then $\rho\pi \in \mathrm{Aut}(G)$.*

Proof. $G^{\rho\pi} = G^{\rho\pi} \times H^{\rho\pi} = (G \times H)^{\rho\pi} = (G \times H)^\pi = G$. If $g \in G$ is such that $g^{\rho\pi} = 1$ then $g^\rho \in H$, and so $g \in H^{\rho^{-1}} = H$, whence $g = 1$. ∎

Theorem 5.2. $\mathrm{Aut}(\Gamma_n)$ *is a split extension of* $\mathrm{Inn}(\Gamma_n)$ *by* H_n.

Proof. We will show by induction on n that $\mathrm{Aut}(\Gamma_n)$ is generated by $\mathrm{Inn}(\Gamma_n)$ and H_n. The cases $n \leqslant 1$ are trivial, and the case $n=2$ was proved by Jones and Thornton [6]. We now assume that $n>2$ and that $\mathrm{Aut}(\Gamma_m) = \langle \Gamma_m, H_m \rangle$ for all $m<n$.

Let $\alpha \in \mathrm{Aut}(\Gamma_n)$. We will compose α with elements of $\langle \Gamma_n, H_n \rangle$ to produce the trivial automorphism. This will show that $\alpha \in \langle \Gamma_n, H_n \rangle$. By Proposition 4.7 there exists $\alpha_1 \in \langle \Gamma_n, H_n \rangle$ such that $r_n^{\alpha\alpha_1} = r_n$. Thus $\alpha\alpha_1$ restricts to an automorphism of $C_{\Gamma_n}(r_n) = \Gamma_{n-2} \times \langle r_n \rangle$ and so, by Proposition 5.1, if π is the natural epimorphism $\pi\colon \Gamma_{n-2} \times \langle r_n \rangle \to \Gamma_{n-2}$ then $\alpha\alpha_1\pi \in \mathrm{Aut}(\Gamma_{n-2})$. Thus, by the inductive hypothesis, there exist $h \in H_{n-2}$ and $g \in \Gamma_{n-2}$ such that $r_i^{\alpha\alpha_1} = r_i^{hg} z_i$ for some $z_i \in \langle r_n \rangle$, $0 \leqslant i \leqslant n-2$. If $\psi_{n-2} \in H_{n-2}$ is defined by $r_i \mapsto r_i$ for $0 \leqslant i \leqslant n-2$, $i \neq n-4$, and $r_{n-4} \mapsto r_{n-4} r_{n-2}$, one has $H_{n-2} = \langle \theta_{n-2} \rangle \langle \psi_{n-2} \rangle \langle \phi_{n-2} \rangle$. Let $\beta \in \langle \theta_{n-2} \rangle$, $\gamma \in \langle \psi_{n-2} \rangle$, $\delta \in \langle \phi_{n-2} \rangle$ be such that $h = \beta\gamma\delta$. Note that ϕ_{n-2} is the restriction of ϕ_n to Γ_{n-2}. Define $\delta' \in H_n$ by $\delta' = 1$ if $\delta = 1$ else $\delta' = \phi_n$. Let $\alpha_2 = g^{-1}\delta'$. Then $r_i^{\alpha\alpha_1\alpha_2} = r_i^{\beta\gamma} z_i$ for $0 \leqslant i \leqslant n-2$, and $r_n^{\alpha\alpha_1\alpha_2} = r_n$.

If $i \notin \{2, n-2, n-1, n\}$ then $|ZC_{\Gamma_n}(r_i)| \lneqq |ZC_{\Gamma_n}(r_i^{\beta\gamma} r_n)|$, whence $z_i = 1$ for $i \notin \{2, n-2\}$. Similar arguments show that $\alpha\alpha_1\alpha_2$ in fact fixes r_i for

$0 \leqslant i \leqslant n-3$ and can only act on r_{n-2} as multiplication by some $z_{n-2} \in \langle r_n \rangle$. Thus, if we write $z_{n-2} = r_n^{\varepsilon}$ and let $\rho = \alpha\alpha_1\alpha_2\psi_n^{\varepsilon}$ then ρ fixes r_i for $i \neq n-1$.

Now ρ restricts to an automorphism of $C_{\Gamma_n}(r_{n-3}r_{n-5}r_{n-7}\cdots)$. If we let K be the subgroup $\langle r_n, r_{n-1} \rangle$ and L the subgroup $\langle r_{n-3}, r_{n-5}, r_{n-7}, \ldots \rangle$ then $C_{\Gamma_n}(r_{n-3}r_{n-5}r_{n-7}\cdots) = K \times L$ and $L^{\rho} = L$. So, by Proposition 5.1, if π is the natural epimorphism π: $K \times L \to K$ then $\rho\pi \in \mathrm{Aut}(K)$. The set $\{r_{n-1}r_n, r_nr_{n-1}\}$ is characteristic in K and $r_n^{\rho\pi} = r_n$; thus there exists $z \in \langle r_n \rangle$ such that $r_{n-1}^{\rho\pi} = r_{n-1}^{z}$. Whence ρz multiplies r_{n-1} by some $w \in L$ and fixes all other r_i.

If $n=3$ then there exists $\eta \in \{0, 1\}$ such that $w = r_0^{\eta}$, whence $\rho z \phi_n^{\eta} = 1$ and we are done. If $n > 3$ then $ZC_{\Gamma_n}(r_{n-1}) \cong C_2$. Thus $ZC_{\Gamma_n}(r_{n-1}w) \cong C_2$ and so $w = 1$ and we are done. ∎

Acknowledgments

The author thanks G. A. Jones for suggesting the problem and N. Biggs and D. Singerman for helpful comments on presentation. She also thanks the SERC of the United Kingdom for their financial support during the preparation of [5], where the general result first appeared. This paper was written during the author's tenureship of an IBM (UK) Fellowship.

References

1. N. Bourbaki, "Groupes et algebres de Lie," Chap. 4–6, Eléments de Mathématique, Vol. 34, Hermann, Paris, 1968.
2. R. P. Bryant and D. Singerman, Foundations of the theory of maps on surfaces with boundary, *Quart. J. Math. Oxford* (2) **36** (1985), 17–41.
3. A. W. M. Dress, Regular polytopes and equivariant tessellations from a combinatorial point of view, *in* "Algebraic Topology Göttingen 1984," pp. 56–72, Lecture Notes in Mathematics, Vol. 1172, Springer-Verlag, Berlin, 1985.
4. C. Gagliardi, A combinatorial characterization of 3-manifold crystalizations, *Boll. Un. Mat. Ital. A* **16A** (1979), 441–449.
5. L. D. James, "Maps and Hypermaps—Operations and Symmetry," Ph. D. thesis, Southampton University, 1985.
6. G. A. Jones and J. S. Thornton, Operations on maps and outer automorphisms, *J. Combin. Theory Ser. B* **35** (1984), 93–103.
7. S. Lins, Graph-encoded maps, *J. Combin. Theory Ser. B* **32** (1982), 171–181.
8. R. C. Lyndon and P. E. Schupp, "Combinatorial Group Theory," Springer-Verlag, Berlin/Heidelberg/New York, 1977.
9. M. A. Ronan, Coverings and automorphisms of chamber systems, *European J. Combin.* **1** (1980), 259–269.
10. J. Tits, A local approach to buildings, *in* "The Geometric Vein" (C. Davis, B. Grümbaum, and F. A. Sherk, Eds.), Springer-Verlag, New York/Heidelberg/Berlin, 1981.
11. A. Vince, Combinatorial maps, *J. Combin. Theory Ser. B* **34** (1983), 1–21.
12. S. E. Wilson, Operators over regular maps, *Pacific J. Math.* **81** (1979), 559–568.